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ARCH and GARCH Models: Quasi-Likelihood and Asymptotic Quasi-Likelihood Approaches

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Abstract

This chapter considers estimation of autoregressive conditional heteroscedasticity (ARCH) and the generalized autoregressive conditional heteroscedasticity (GARCH) models using quasi-likelihood (QL) and asymptotic quasi-likelihood (AQL) approaches. The QL and AQL estimation methods for the estimation of unknown parameters in ARCH and GARCH models are developed. Distribution assumptions are not required of ARCH and GARCH processes by QL method. Nevertheless, the QL technique assumes knowing the first two moments of the process. However, the AQL estimation procedure is suggested when the conditional variance of process is unknown. The AQL estimation substitutes the variance and covariance by kernel estimation in QL. Reports of simulation outcomes, numerical cases, and applications of the methods to daily exchange rate series and weekly prices' changes of crude oil are presented.

Keywords: ARCH model, GARCH model, the quasi-likelihood, asymptotic quasi-likelihood, martingale difference, daily exchange rate series, prices changes of crude oil

1. Introduction

The autoregressive conditional heteroscedasticity (ARCH(q)) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (1)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \dots + \alpha_q \xi_{t-q}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \quad (2)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. For estimation and applications of ARCH models, see [1–19]. Moreover, ARCH models have now become the standard textbook material in econometrics and finance as exemplified by, for example, [20–23].

The generalized autoregressive conditional heteroscedasticity (GARCH(p,q)) process y_t is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (3)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \cdots + \alpha_p \xi_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (4)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$.

The GARCH model was developed by Bollerslev [24] to extend the earlier work on ARCH models by Engle [1]. For estimation and applications of GARCH models, (see, [2, 3, 6–8, 10, 11, 14]). Moreover, GARCH models have now become the standard textbook material in econometrics and finance as exemplified by, for example, [20–23].

This chapter considers estimation of ARCH and GARCH models using quasi-likelihood (QL) and asymptotic quasi-likelihood (AQL) approaches. Distribution assumptions are not required of ARCH and GARCH processes by the QL method. But, the QL technique assumes knowing the first two moments of the process. However, The AQL estimation procedure is suggested when the conditional variance of process is unknown. The AQL estimation substitutes the variance and covariance by kernel estimation in QL.

This chapter is structured as follows. Section 2 introduces the QL and AQL approaches. The estimation of ARCH model using QL and AQL methods are developed in Section 3. The estimation of GARCH model using QL and AQL methods are developed in Section 4. Reports of simulation outcomes, numerical cases and applications of the methods to a daily exchange rate series, and weekly prices changes of crude oil are also presented. Summary and conclusion are given in Section 5.

2. The QLE and AQL methods

Let the observation equation be given by

$$\mathbf{y}_t = \mathbf{f}_t(\theta) + \zeta_t, \quad t = 1, 2, 3, \dots, T, \quad (5)$$

where ζ_t is a sequence of martingale difference with respect to \mathcal{F}_t , \mathcal{F}_t denotes the σ -field generated by $\mathbf{y}_t, \mathbf{y}_{t-1}, \dots, \mathbf{y}_1$ for $t \geq 1$; that is, $E(\zeta_t | \mathcal{F}_{t-1}) = E_{t-1}(\zeta_t) = 0$, where $\mathbf{f}_t(\theta)$ is an \mathcal{F}_{t-1} measurable and θ is parameter vector, which belongs to an open subset $\Theta \in \mathbb{R}^d$. Note that θ is a parameter of interest.

2.1 The QL method

For the model given by Eq. (5), assume that $E_{t-1}(\zeta_t \zeta_t') = \Sigma_t$ is known. Now, the linear class \mathcal{G}_T of the estimating function (EF) can be defined by

$$\mathcal{G}_T = \left\{ \sum_{t=1}^T \mathbf{W}_t (\mathbf{y}_t - \mathbf{f}_t(\theta)) \right\}$$

and the quasi-likelihood estimation function (QLEF) can be defined by

$$\mathbf{G}_T^*(\theta) = \sum_{t=1}^T \dot{\mathbf{f}}_t(\theta) \Sigma_t^{-1} (\mathbf{y}_t - \mathbf{f}_t(\theta)) \quad (6)$$

where \mathbf{W}_t is \mathcal{F}_{t-1} -measureable and $\dot{\mathbf{f}}_t(\theta) = \partial \mathbf{f}_t(\theta) / \partial \theta$. Then, the estimation of θ by the QL method is the solution of the QL equation $\mathbf{G}_T^*(\theta) = 0$ (see [25]).

If the sub-estimating function spaces of \mathcal{G}_T are considered as follows:

$$\mathcal{G}_t = \{\mathbf{W}_t(\mathbf{y}_t - \mathbf{f}_t(\theta))\}$$

then the QLEF can be defined by

$$\mathbf{G}_{(t)}^*(\theta) = \dot{\mathbf{f}}_t(\theta) \Sigma_t^{-1} (\mathbf{y}_t - \mathbf{f}_t(\theta)) \quad (7)$$

and the estimation of θ by the QL method is the solution of the QL equation $\mathbf{G}_{(t)}^*(\theta) = 0$.

A limitation of the QL method is that the nature of Σ_t may not be obtainable. A misidentified Σ_t could result in a deceptive inference about parameter θ . In the next subsection, we will introduce the AQL method, which is basically the QL estimation assuming that the covariance matrix Σ_t is unknown.

2.2 The AQL method

The QLEF (see Eqs. (6) and (7)) relies on the information of Σ_t . Such information is not always accessible. To find the QL when $E_{t-1}(\zeta_t \zeta_t')$ is not accessible, Lin [26] proposed the AQL method.

Definition 2.2.1: Let $\mathbf{G}_{T,n}^*$ be a sequence of the EF in \mathcal{G} . For all $\mathbf{G}_T \in \mathcal{G}$, if

$$(E\dot{\mathbf{G}}_T)^{-1} (E\mathbf{G}_T \mathbf{G}_T')' (E\dot{\mathbf{G}}_T)^{-1} - (E\dot{\mathbf{G}}_{T,n}^*)^{-1} (E\mathbf{G}_{T,n}^* \mathbf{G}_{T,n}^{*'})' (E\dot{\mathbf{G}}_{T,n}^*)^{-1}$$

is asymptotically nonnegative definite, $\mathbf{G}_{T,n}^*$ can be denoted as the asymptotic quasi-likelihood estimation function (AQLEF) sequence in \mathcal{G} , and the AQL sequence estimate $\theta_{T,n}$ by the AQL method is the solution of the AQL equation $\mathbf{G}_{T,n}^* = 0$.

Suppose, in probability, $\Sigma_{t,n}$ is converging to $E_{t-1}(\zeta_t \zeta_t')$. Then,

$$\mathbf{G}_{T,n}^*(\theta) = \sum_{t=1}^T \dot{\mathbf{f}}_t(\theta) \Sigma_{t,n}^{-1} (\mathbf{y}_t - \mathbf{f}_t(\theta)) \quad (8)$$

expresses an AQLEF sequence. The solution of $\mathbf{G}_{T,n}^*(\theta) = 0$ expresses the AQL sequence estimate $\{\theta_{T,n}^*\}$, which converges to θ under certain regular conditions.

In this chapter, the kernel smoothing estimator of Σ_t is suggested to find $\Sigma_{t,n}$ in the AQLEF (Eq. (8)). A wide-ranging appraisal of the Nadaraya-Watson (NW) estimator-type kernel estimator is available in [27]. By using these kernel estimators, the AQL equation becomes

$$\mathbf{G}_{T,n}^*(\theta) = \sum_{t=1}^T \dot{\mathbf{f}}_t(\theta) \hat{\Sigma}_{t,n}^{-1} (\hat{\theta}^{(0)}) (\mathbf{y}_t - \mathbf{f}_t(\theta)) = 0. \quad (9)$$

The estimation of θ by the AQL method is the solution to Eq. (9). Iterative techniques are suggested to solve the AQL equation (Eq. (9)). Such techniques start with the ordinary least squares (OLS) estimator $\hat{\theta}^{(0)}$ and use $\hat{\Sigma}_{t,n}(\hat{\theta}^{(0)})$ in the AQL equation (Eq. (9)) to obtain the AQL estimator $\hat{\theta}^{(1)}$. Repeat this a few times until it converges.

For estimation of unknown parameters in fanatical models by QL and AQL approaches, see [21, 28–33]. The next sections present the parameter estimation of ARCH model using the QL and AQL methods.

3. Parameter estimation of ARCH(q) model using the QL and AQL methods

In this section, we will develop the estimation of ARCH model using QL and AQL methods.

3.1 Parameter estimation of ARCH(q) model using the QL method

The ARCH(q) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (10)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \dots + \alpha_q \xi_{t-q}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \quad (11)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. For this scenario, the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_q \xi_{t-q}^2 \end{pmatrix}.$$

The QLEF to estimate σ_t^2 is given by

$$\begin{aligned} G_{(t)}(\sigma_t^2) &= (0, 1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_q \xi_{t-q}^2 \end{pmatrix} \\ &= \sigma_\zeta^{-2} (\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_q \xi_{t-q}^2). \end{aligned} \quad (12)$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q, \sigma_{\zeta_0}^2)$ and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$:

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \dots + \alpha_q \hat{\xi}_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (13)$$

The QLEF, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, to estimate the parameters $\mu, \alpha_0, \alpha_1, \dots, \alpha_q$ is given by

$$\begin{aligned} G_T(\mu, \alpha_0, \alpha_1, \dots, \alpha_q) &= \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\hat{\xi}_{t-q}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \hat{\xi}_{t-1}^2 - \dots - \alpha_q \hat{\xi}_{t-q}^2 \end{pmatrix}. \end{aligned}$$

The QL estimate of $\mu, \alpha_0, \alpha_1, \dots, \alpha_q$ is the solution of $G_T(\mu, \alpha_0, \alpha_1, \dots, \alpha_q) = 0$, where $\hat{\xi}_t = \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \dots - \hat{\alpha}_q \hat{\xi}_{t-q}^2, t = 1, 2, 3, \dots, T$ and

$$\hat{\sigma}_\xi^2 = \frac{\sum_{t=1}^T (\hat{\xi}_t - \bar{\xi})^2}{T-1} \quad (14)$$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_q, \hat{\sigma}_\xi^2)$ is an initial value in the iterative procedure.

3.2 Parameter estimation of ARCH(q) model using the AQL method

For ARCH(q) model given by Eqs. (10) and (11) and using the same argument listed under Eq. (11). First, to estimate σ_t^2 , so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0, 1) \Sigma_{t,n}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_q \xi_{t-q}^2 \end{pmatrix}$$

Given $\hat{\xi}_0 = 0, \theta_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q), \Sigma_{t,n}^{(0)} = \mathbf{I}_2$, and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \dots + \alpha_q \hat{\xi}_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (15)$$

Second, by kernel estimation method, we find

$$\hat{\Sigma}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t, \sigma_t) \\ \hat{\sigma}_n(\sigma_t, y_t) & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Third, to estimate the parameters $\theta_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q)$ using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of (AQLEF):

$$G_T(\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-q}^2 \end{pmatrix} \hat{\Sigma}_{t,n}^{-1} \times \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_q \xi_{t-q}^2 \end{pmatrix}.$$

The AQL estimate of $\theta_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q)$ is the solution of $G_T(\theta_0) = 0$. The estimation procedure will be iteratively repeated until it converges.

3.3 Simulation studies for the ARCH(1) model

The estimation of ARCH(1) model using QL and AQL methods are considered in simulation studies. The ARCH(1) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (16)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \quad (17)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_\xi^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.

3.3.1 Parameter estimation of ARCH(1) model using the QL method

For ARCH(1) given by Eqs. (16) and (17), the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}.$$

The QLEF to estimate σ_t^2 is given by

$$\begin{aligned} G_{(t)}(\sigma_t^2) &= (0, 1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix} \\ &= \sigma_\zeta^{-2} (\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2). \end{aligned} \quad (18)$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_0, \alpha_1, \sigma_{\xi_0}^2)$ and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2, \quad t = 1, 2, 3, \dots, T. \quad (19)$$

To estimate the parameters μ , α_0 , and α_1 , using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, the QLEF is given by

$$G_T(\mu, \alpha_0, \alpha_1) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\xi_0}^2 \end{pmatrix}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \hat{\xi}_{t-1}^2 \end{pmatrix}.$$

The solution of $G_T(\mu, \alpha_0, \alpha_1) = 0$ is the QL estimate of μ , α_0 , and α_1 . Therefore

$$\hat{\mu} = \sum_{t=1}^T \frac{y_t}{\hat{\sigma}_t^2} / \sum_{t=1}^T \frac{1}{\hat{\sigma}_t^2}. \quad (20)$$

$$\hat{\alpha}_1 = \frac{T \sum_{t=1}^T \hat{\sigma}_t^2 \hat{\xi}_{t-1}^2 - \sum_{t=1}^T \hat{\sigma}_t^2 \sum_{t=1}^T \hat{\xi}_{t-1}^2}{T \sum_{t=1}^T \hat{\xi}_{t-1}^4 - \left(\sum_{t=1}^T \hat{\xi}_{t-1}^2 \right)^2}. \quad (21)$$

$$\hat{\alpha}_0 = \frac{\sum_{t=1}^T \hat{\sigma}_t^2 - \hat{\alpha}_1 \sum_{t=1}^T \hat{\xi}_{t-1}^2}{T}. \quad (22)$$

and let

$$\hat{\sigma}_\zeta^2 = \frac{\sum_{t=1}^T (\hat{\xi}_t - \bar{\hat{\xi}})^2}{T - 1} \quad (23)$$

where $\hat{\xi}_t = \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2$, $t = 1, 2, 3, \dots, T$.

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\sigma}_\zeta^2)$ is an initial value in the iterative procedure.

The initial values might be affected the estimation results. For extensive discussion on assigning initial values in the QL estimation procedures, see [21, 34].

3.3.2 Parameter estimation of ARCH(1) model using the AQL method

Considering the ARCH(1) model given by Eqs. (16) and (17) and using the same argument listed under Eq. (17). First, we need to estimate σ_t^2 , so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0, 1) \Sigma_{t,n}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}$$

Given $\hat{\xi}_0 = 0$, $\theta_0 = (\mu_0, \alpha_0, \alpha_1, \mu_0)$, $\Sigma_{t,n}^{(0)} = \mathbf{I}_2$ and $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2, \quad t = 1, 2, 3 \dots, T. \quad (24)$$

Second, by kernel estimation method, we find

$$\hat{\Sigma}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t, \sigma_t) \\ \hat{\sigma}_n(\sigma_t, y_t) & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Third, to estimate the parameters $\theta = (\mu, \alpha_0, \alpha_1)$ using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of AQLEF:

$$G_T(\mu, \alpha_0, \alpha_1) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1} \end{pmatrix} \hat{\Sigma}_{t,n}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 \end{pmatrix}.$$

The AQL estimate of γ , ϕ , and μ is the solution of $G_T(\mu, \alpha_0, \alpha_1) = 0$. Therefore

$$\hat{\mu} = \sum_{t=1}^T \frac{y_t}{\hat{\sigma}_n(y_t)} / \sum_{t=1}^T \frac{1}{\hat{\sigma}_n(y_t)}. \quad (25)$$

$$\hat{\alpha}_1 = \frac{\left(\sum_{t=1}^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right) - \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right)}{\left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right)^2 - \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)} \right)}. \quad (26)$$

$$\hat{\alpha}_0 = \frac{\left(\sum_{t=1}^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)} \right) - \hat{\alpha}_1 \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right)}{\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)}}. \quad (27)$$

and let

$$\hat{\sigma}_\zeta^2 = \frac{\sum_{t=1}^T \left(\hat{\xi}_t - \bar{\xi} \right)^2}{T - 1} \quad (28)$$

The estimation procedure will be iteratively repeated until it converges.

For each parameter setting, $T = 500$ observations are simulated from the true model. We then replicate the experiment for 1000 times to obtain the mean and root mean squared errors (RMSE) for $\hat{\alpha}_0$, $\hat{\alpha}_1$, and $\hat{\mu}$. In **Table 1**, QL denotes the QL estimate and AQL denotes the AQL estimate.

We generated $N = 1000$ independent random samples of size $T = 20, 40, 60, 80$, and 100 from ARCH(1) model. In **Table 2**, the QL and AQL estimation methods show the property of consistency, the RMSE decreases as the sample size increases.

3.4 Empirical applications

The first data set we analyze are the daily exchange rate of $r_t = AUD/USD$ (Australian dollar/US dollar) for the period from 5/6/2010 to 5/5/2016, 1590 observations in total. The ARCH model (Eqs. (16) and (17)) is used to model $y_t = \log(r_t) - \log(r_{t-1})$.

We used the S + FinMetrics function archTest to carry out Lagrange multiplier (ML) test for the presence of ARCH effects in the residuals (see [35]). For r_t the p-values are significant (< 0.05 level), so reject the null hypothesis that there are no ARCH effects and we fit $\{y_t\}$ by following models:

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \tag{29}$$

	α_0	α_1	μ	α_0	α_1	μ	α_0	α_1	μ
True	0.010	0.980	1.30	0.010	0.980	-1.30	0.010	0.980	0.030
QL	0.009	0.989	1.299	0.009	0.989	-1.30	0.009	0.989	0.029
	0.001	0.010	0.006	0.001	0.010	0.006	0.001	0.010	0.006
AQL	0.009	0.989	1.30	0.009	0.989	-1.29	0.009	0.989	0.030
	0.001	0.010	0.0003	0.002	0.009	0.0003	0.001	0.009	0.0003
True	0.050	0.950	1.30	0.050	0.950	-1.30	0.050	.950	0.030
QL	0.049	0.949	1.29	0.049	0.940	-1.30	0.049	0.94	0.029
	0.001	0.0001	0.014	0.001	0.010	0.014	0.001	0.010	0.014
AQL	0.049	0.940	1.32	0.049	0.940	-1.30	0.049	0.940	0.032
	0.001	0.010	0.018	0.001	0.010	0.018	0.001	0.01	0.001
True	0.10	0.90	1.30	0.10	0.90	-1.30	0.10	0.90	0.030
QL	0.098	0.910	1.29	0.098	0.910	-1.30	0.098	0.910	0.023
	0.002	0.010	0.019	0.002	0.010	0.020	0.002	0.010	0.029
AQL	0.098	0.910	1.31	0.098	0.910	-1.32	0.098	0.910	0.031
	0.002	0.010	0.012	0.002	0.010	0.021	0.001	0.010	0.001
True	0.1	0.90	-0.03	0.05	0.95	-0.03	0.01	0.98	-0.03
QL	0.098	0.910	-0.031	0.051	0.949	-0.030	0.009	0.990	-0.030
	0.002	0.010	0.019	0.001	0.001	0.014	0.001	0.016	0.006
AQL	0.098	0.910	-0.031	0.051	0.949	-0.031	0.009	0.990	-0.031
	0.002	0.010	0.001	0.001	0.001	0.002	0.001	0.010	0.001

Table 1.
The QL and AQL estimates and the RMSE of each estimate is stated below that estimate for ARCH model.

		α_0	α_1	μ	α_0	α_1	μ
T = 20	True	0.010	0.980	−0.030	0.05	0.950	1.3
	QL	0.009	0.990	−0.029	0.0495	0.9485	1.300
		0.0008	0.0100	0.0319	0.0005	0.0015	0.0703
	AQL	0.009	0.990	−0.031	0.0495	0.9485	1.3107
		0.0009	0.010	0.0084	0.0005	0.0015	0.0213
	QL	0.009	0.990	−0.031	0.0495	0.9485	1.3015
T = 40	QL	0.00089	0.010	0.0223	0.0005	0.0015	0.0492
		0.009	0.990	−0.031	0.0495	0.9485	1.3113
	AQL	0.00089	0.010	0.0039	0.0005	0.0015	0.0143
		0.009	0.990	−0.029	0.0495	0.9485	1.300
	QL	0.0009	0.010	0.0180	0.0005	0.0015	0.0404
		0.009	0.990	−0.031	0.0495	0.9485	1.311
T = 60	QL	0.0009	0.010	0.0027	0.0005	0.0015	0.0128
		0.009	0.990	−0.029	0.0490	0.9485	1.300
	AQL	0.0009	0.010	0.016	0.0005	0.0015	0.0353
		0.009	0.990	−0.310	0.0495	0.9485	1.3112
	QL	0.0009	0.010	0.0020	0.0005	0.0015	0.0119
		0.009	0.990	0.0292	0.0495	0.9485	1.3017
T = 80	QL	0.0009	0.010	0.0142	0.0005	0.0015	0.0314
		0.009	0.990	−0.031	0.0495	0.9485	1.3111
	AQL	0.0009	0.010	0.0018	0.0005	0.0015	0.0116
		0.009	0.990	−0.029	0.0495	0.9485	1.300
	QL	0.0009	0.010	0.016	0.0005	0.0015	0.0353
		0.009	0.990	−0.310	0.0495	0.9485	1.3112
T = 100	QL	0.0009	0.010	0.0020	0.0005	0.0015	0.0119
		0.009	0.990	0.0292	0.0495	0.9485	1.3017
	AQL	0.0009	0.010	0.0142	0.0005	0.0015	0.0314
		0.009	0.990	−0.031	0.0495	0.9485	1.3111
	QL	0.0009	0.010	0.0018	0.0005	0.0015	0.0116
		0.009	0.990	−0.029	0.0495	0.9485	1.300

Table 2.
The QL and AQL estimates and the RMSE of each estimate is stated below that estimate for ARCH model with different sample size.

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \tag{30}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_\xi^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.

The estimation of unknown parameters, $(\alpha_0, \alpha_1, \mu)$, using QL and AQL are given in **Table 3**. Conclusion can be drawn based on the standardized residuals from the fourth column in **Table 3**, which favors the QL method, gives smaller standardized residuals, better than AQL method.

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\mu}$	$\frac{\bar{\xi}_t}{s.d(\xi_t)}$
QL	0.1300	0.8387	−0.00012	0.00013
AQL	0.0200	0.9599	−0.00111	0.1350

Table 3.
Estimation of α_0, α_1, μ for the exchange rate pound/dollar data.

4. Parameter estimation of GARCH(p,q) model using the QL and AQL methods

In this section, we developing the estimation of GARCH model using QL and AQL methods.

4.1 Parameter estimation of GARCH(p,q) model using the QL method

The GARCH(p,q) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (31)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \dots + \alpha_p \xi_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_q \sigma_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (32)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. For this scenario, the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_p \xi_{t-p}^2 - \beta_1 \sigma_{t-1}^2 - \dots - \beta_q \sigma_{t-q}^2 \end{pmatrix}.$$

The QLEF to estimate σ_t^2 is given by

$$\begin{aligned} G_{(t)}(\sigma_t^2) &= (0, 1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} \\ &= \sigma_\zeta^{-2} \left(\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \dots - \alpha_p \xi_{t-p}^2 - \beta_1 \sigma_{t-1}^2 - \dots - \beta_q \sigma_{t-q}^2 \right). \end{aligned} \quad (33)$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, \sigma_{\zeta_0}^2)$, $\hat{\xi}_{t-i}^2 = (y_{t-i} - \mu_0)^2$, and $\hat{\sigma}_{t-j}^2$ are the QL estimations of σ_{t-j}^2 , where $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \dots + \alpha_p \hat{\xi}_{t-p}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \dots + \beta_q \hat{\sigma}_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (34)$$

The QLEF, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, to estimate the parameters $\theta = \mu, \alpha_0, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_q$ is given by

$$G_T(\theta) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\hat{\xi}_{t-p}^2 \\ 0 & -\hat{\sigma}_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\hat{\sigma}_{t-q}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}$$

The QL estimate of $\mu, \alpha_0, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_q$ is the solution of $G_T(\theta) = 0$, where $\hat{\xi}_t = \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \dots - \hat{\alpha}_p \hat{\xi}_{t-p}^2 - \hat{\beta}_1 \hat{\sigma}_{t-1}^2 - \dots - \hat{\beta}_q \hat{\sigma}_{t-q}^2, t = 1, 2, 3, \dots, T$ and

$$\hat{\sigma}_\xi^2 = \frac{\sum_{t=1}^T (\hat{\xi}_t - \bar{\xi})^2}{T-1} \quad (35)$$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_p, \hat{\beta}_1, \dots, \hat{\beta}_q, \hat{\sigma}_\xi^2)$ is an initial value in the iterative procedure.

4.2 Parameter estimation of GARCH(p,q) model using the AQL method

Considering the GARCH(p,q) model given by Eqs. (31) and (32) and using the same argument listed under Eq. (32). First, we need to estimate σ_t^2 , so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0, 1) \Sigma_{t,n}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}$$

Given $\hat{\xi}_0 = 0, \theta_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q), \Sigma_{t,n}^{(0)} = \mathbf{I}_2$, and $\hat{\xi}_{t-i}^2 = (y_{t-i} - \mu_0)^2$, and $\hat{\sigma}_{t-j}^2$ is the AQL estimation of σ_{t-j}^2 , where $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \dots + \alpha_p \hat{\xi}_{t-p}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \dots + \beta_q \hat{\sigma}_{t-q}^2, \quad t = 1, 2, 3, \dots, T. \quad (36)$$

Second, by kernel estimation method, we find

$$\hat{\Sigma}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t, \sigma_t) \\ \hat{\sigma}_n(\sigma_t, y_t) & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Third, to estimate the parameters $\theta_0 = (\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q)$ using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of (AQLEF):

$$G_T(\mu_0, \alpha_0, \alpha_1, \dots, \alpha_q) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-q}^2 \\ 0 & -\sigma_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\sigma_{t-q}^2 \end{pmatrix} \hat{\Sigma}_{t,n}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}.$$

The AQL estimate of $\theta = (\mu, \alpha_0, \alpha_1, \dots, \alpha_q)$ is the solution of $G_T(\theta) = 0$. The estimation procedure will be iteratively repeated until it converges.

4.3 Simulation studies for the GARCH(1,1) model

The estimation of GARCH(1,1) model using QL and AQL methods are considered in simulation studies. The GARCH(1,1) process is defined by

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \quad (37)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \quad (38)$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.

4.3.1 Parameter estimation of GARCH(1,1) model using the QL method

For GARCH(1,1) given by Eqs. (37) and (38), the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix}.$$

The QLEF to estimate σ_t^2 is given by

$$\begin{aligned} G_{(t)}(\sigma_t^2) &= (0, 1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix} \\ &= \sigma_\zeta^{-2} (\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2). \end{aligned} \quad (39)$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_0, \alpha_1, \sigma_{\xi_0}^2)$, $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, and $\hat{\sigma}_{t-1}^2$ is the QL estimation of σ_{t-1}^2 , then the QL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2, \quad t = 1, 2, 3, \dots, T. \quad (40)$$

To estimate the parameters μ , α_0 , and α_1 , using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, the QLEF is given by

$$\begin{aligned} G_T(\mu, \alpha_0, \alpha_1, \beta_1) &= \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1}^2 \\ 0 & -\hat{\sigma}_{t-1}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\xi_0}^2 \end{pmatrix}^{-1} \\ &\quad * \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \hat{\xi}_{t-1}^2 - \beta_1 \hat{\sigma}_{t-1}^2 \end{pmatrix}. \end{aligned}$$

The solution of $G_T(\mu, \alpha_0, \alpha_1, \beta_1) = 0$ is the QL estimate of μ , α_0 , α_1 , and β_1 . Therefore

$$\hat{\mu} = \sum_{t=1}^T \frac{y_t}{\hat{\sigma}_t^2} / \sum_{t=1}^T \frac{1}{\hat{\sigma}_t^2}. \quad (41)$$

$$\hat{\beta}_1 = \frac{S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2} S_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2} S_{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2}}{S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}^2 - S_{\hat{\sigma}_{t-1}^2 \hat{\sigma}_{t-1}^2} S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \quad (42)$$

$$\hat{\alpha}_1 = \frac{S_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - \hat{\beta}_1 S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}}{S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \quad (43)$$

$$\hat{\alpha}_0 = \frac{\sum_{t=1}^T \hat{\sigma}_t^2 - \hat{\alpha}_1 \sum_{t=1}^T \hat{\xi}_{t-1}^2 - \hat{\beta}_1 \sum_{t=1}^T \hat{\sigma}_{t-1}^2}{T}. \quad (44)$$

and let

$$\hat{\sigma}_{\xi}^2 = \frac{\sum_{t=1}^T (\hat{\xi}_t - \bar{\xi})^2}{T-1} \quad (45)$$

where

$$\begin{aligned} \hat{\xi}_t &= \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \hat{\beta}_1 \hat{\sigma}_{t-1}^2, \quad t = 1, 2, 3, \dots, T, \\ S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2} &= \sum_{t=1}^T \hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2 - \frac{\sum_{t=1}^T \hat{\sigma}_{t-1}^2 \sum_{t=1}^T \hat{\xi}_{t-1}^2}{T}, \\ S_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} &= \sum_{t=1}^T \hat{\sigma}_t^2 \hat{\xi}_{t-1}^2 - \frac{\sum_{t=1}^T \hat{\sigma}_t^2 \sum_{t=1}^T \hat{\xi}_{t-1}^2}{T}, \\ S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2} &= \sum_{t=1}^T \hat{\xi}_{t-1}^4 - \frac{\left(\sum_{t=1}^T \hat{\xi}_{t-1}^2\right)^2}{T}, \\ S_{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2} &= \sum_{t=1}^T \hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2 - \frac{\sum_{t=1}^T \hat{\sigma}_t^2 \sum_{t=1}^T \hat{\sigma}_{t-1}^2}{T}, \\ S_{\hat{\sigma}_{t-1}^2 \hat{\sigma}_{t-1}^2} &= \sum_{t=1}^T \hat{\sigma}_{t-1}^4 - \frac{\left(\sum_{t=1}^T \hat{\sigma}_{t-1}^2\right)^2}{T}. \end{aligned}$$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\sigma}_{\xi}^2)$ is an initial value in the iterative procedure.

The initial values might be affected the estimation results. For extensive discussion on assigning initial values in the QL estimation procedures, see [21, 34].

4.3.2 Parameter estimation of GARCH(1,1) model using the AQL method

Considering the GARCH(1,1) model given by Eqs. (37) and (38) and using the same argument listed under (Eq. (38)). First, we need to estimate σ_t^2 , so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0, 1) \Sigma_{t,n}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \hat{\xi}_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix}$$

Given $\hat{\xi}_0 = 0$, $\theta_0 = (\mu_0, \alpha_{0,0}, \alpha_{1,0}, \beta_{1,0})$, $\Sigma_{t,n}^{(0)} = \mathbf{I}_2$, $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, and $\hat{\sigma}_{t-1}^2$ is the AQL estimation of σ_{t-1}^2 , then the AQL estimation of σ_t^2 is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2, \quad t = 1, 2, 3, \dots, T. \quad (46)$$

Second, by kernel estimation method, we find

$$\hat{\Sigma}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & 0 \\ 0 & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Third, to estimate the parameters $\theta = (\mu, \alpha_0, \alpha_1, \beta_1)$ using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of AQLEF:

$$G_T(\mu, \alpha_0, \alpha_1, \beta_1) = \sum_{t=1}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\hat{\xi}_{t-1}^2 \\ 0 & -\hat{\sigma}_{t-1}^2 \end{pmatrix} \hat{\Sigma}_{t,n}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \hat{\xi}_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix}.$$

The AQL estimate of μ, α_0, α_1 , and β_1 is the solution of $G_T(\mu, \alpha_0, \alpha_1, \beta_1) = 0$. Therefore

$$\hat{\mu} = \sum_{t=1}^T \frac{y_t}{\hat{\sigma}_n(y_t)} / \sum_{t=1}^T \frac{1}{\hat{\sigma}_n(y_t)}. \quad (47)$$

$$\hat{\beta}_1 = \frac{SS_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2} SS_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - SS_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2} SS_{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2}}{SS_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}^2 - SS_{\hat{\sigma}_{t-1}^2 \hat{\sigma}_{t-1}^2} SS_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \quad (48)$$

$$\hat{\alpha}_1 = \frac{SS_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - \hat{\beta}_1 SS_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}}{SS_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \quad (49)$$

$$\hat{\alpha}_0 = \frac{\sum_{t=1}^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)} - \hat{\alpha}_1 \sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} - \hat{\beta}_1 \sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}}{\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)}}, \quad (50)$$

and let

$$\hat{\sigma}_{\bar{\xi}}^2 = \frac{\sum_{t=1}^T (\hat{\xi}_t - \bar{\xi})^2}{T-1} \quad (51)$$

where

$$\begin{aligned} \hat{\xi}_t &= \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \hat{\beta}_1 \hat{\sigma}_{t-1}^2, \quad t = 1, 2, 3, \dots, T, \\ SS_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2} &= \left(\sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) - \left(\sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right), \\ SS_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} &= \left(\sum_{t=1}^T \frac{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) - \left(\sum_{t=1}^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right), \\ SS_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2} &= \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)} \right) - \left(\sum_{t=1}^T \frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right)^2, \\ SS_{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2} &= \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) \sum_{t=1}^T \frac{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} - \sum_{t=1}^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)} \sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}, \\ SS_{\hat{\sigma}_{t-1}^2 \hat{\sigma}_{t-1}^2} &= \left(\sum_{t=1}^T \frac{1}{\hat{\sigma}_n(\sigma_t)} \right) \left(\sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)} \right) - \left(\sum_{t=1}^T \frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)} \right)^2. \end{aligned}$$

	μ	α_0	α_1	β_1	μ	α_0	α_1	β_1
True	0.15	0.65	0.87	0.10	0.20	0.41	0.88	0.08
QL	0.149	0.779	0.865	0.074	0.199	0.461	0.912	0.057
	0.040	0.353	0.011	0.029	0.031	0.155	0.033	0.025
AQL	0.150	0.661	0.851	0.092	0.209	0.405	0.901	0.076
	0.001	0.012	0.019	0.009	0.010	0.006	0.021	0.004
True	-0.10	0.48	0.89	0.08	0.16	0.37	0.9	0.08
QL	-0.101	0.556	0.902	0.058	0.159	0.434	0.922	0.058
	0.034	0.212	0.014	0.024	0.030	0.189	0.024	0.025
AQL	-0.110	0.486	0.891	0.0752	0.161	0.374	0.911	0.076
	0.010	0.006	0.001	0.005	0.001	0.004	0.011	0.004
True	0.18	0.39	0.88	0.08	0.09	0.50	0.89	0.05
QL	0.179	0.447	0.892	0.058	0.089	0.538	0.898	0.036
	0.031	0.146	0.015	0.024	0.033	0.090	0.009	0.015
AQL	0.180	0.395	0.882	0.076	0.091	0.504	0.892	0.046
	0.001	0.005	0.002	0.005	0.002	0.004	0.002	0.004

Table 4.
The QL and AQL estimates and the RMSE of each estimate is stated below that estimate for GARCH model.

The estimation procedure will be iteratively repeated until it converges.
For each parameter setting, $T = 500$ observations are simulated from the true model. We then replicate the experiment for 1000 times to obtain the mean and root mean squared errors (RMSE) for $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\hat{\beta}_1$, and $\hat{\mu}$. In **Table 4**, QL denotes the QL estimate and AQL denotes the AQL estimate.
We generated $N = 1000$ independent random samples of size $T = 20, 40, 60, 80$, and 100 from GARCH(1,1) model. In **Table 5**, The QL and AQL estimation methods show the property of consistency, and the RMSE decreases as the sample size increases.

4.4 Empirical applications

The second set of data is the weekly price changes of crude oil prices P_t . The P_t of Cushing, OK, West Texas Intermediate (US dollars per barrel) is considered for the period from 7/1/2000 to 10/6/2016, with 858 observations in total. The data are transformed into rates of change by taking the first difference of the logs. Thus, $y_t = \log(P_t) - \log(P_{t-1})$ and fit $\{y_t\}$ by using GARCH (1,1):

$$y_t = \mu + \xi_t, \quad t = 1, 2, 3, \dots, T. \tag{52}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \zeta_t, \quad t = 1, 2, 3, \dots, T. \tag{53}$$

ξ_t are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and ζ_t are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$.
The estimation of unknown parameters, $(\alpha_0, \alpha_1, \beta_1, \mu)$, using QL and AQL are given in **Table 6**. Conclusion can be drawn based on the standardized residuals

		μ	α_0	α_1	β_1	μ	α_0	α_1	β_1
	True	0.16	0.37	0.90	0.08	-0.10	0.48	0.89	0.08
	QL	0.17	0.42	0.89	0.07	-0.09	0.51	0.90	0.06
T = 20		0.176	0.511	0.008	0.016	0.169	0.451	0.018	0.022
	AQL	0.16	0.38	0.89	0.07	-0.10	0.47	0.90	0.07
		0.037	0.012	0.007	0.014	0.066	0.014	0.013	0.018
	QL	0.16	0.42	0.89	0.07	-0.09	0.51	0.91	0.06
		0.149	0.422	0.007	0.016	0.137	0.326	0.018	0.021
T = 40	AQL	0.16	0.38	0.89	0.07	-0.10	0.47	0.90	0.07
		0.027	0.012	0.007	0.013	0.022	0.014	0.012	0.016
	QL	0.16	0.42	0.89	0.07	-0.09	0.52	0.91	0.06
		0.121	0.289	0.007	0.018	0.119	0.307	0.018	0.021
	AQL	0.16	0.38	0.89	0.07	-0.10	0.47	0.90	0.07
T = 60		0.019	0.012	0.007	0.011	0.014	0.013	0.012	0.015
	QL	0.16	0.42	0.89	0.07	-0.10	0.51	0.90	0.06
		0.100	0.159	0.007	0.017	0.108	0.248	0.018	0.021
	AQL	0.16	0.38	0.89	0.07	-0.10	0.47	0.90	0.07
		0.012	0.012	0.007	0.011	0.012	0.013	0.012	0.015
T = 80	QL	0.16	0.42	0.89	0.07	-0.10	0.51	0.90	0.06
		0.100	0.159	0.007	0.018	0.101	0.242	0.018	0.021
	AQL	0.16	0.38	0.89	0.07	-0.10	0.47	0.90	0.07
		0.012	0.011	0.007	0.011	0.011	0.013	0.012	0.015

Table 5.
The QL and AQL estimates and the RMSE of each estimate is stated below that estimate for GARCH model with different sample size.

	$\hat{\mu}_0$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\frac{\bar{\xi}_t}{S.d(\xi_t)}$
QL	0.0008	0.566	0.912	0.0004	0.002
AQL	0.0089	0.630	0.972	0.041	0.185

Table 6.
Estimation of $\mu, \alpha_0, \alpha_1, \beta_1$ for the rates of change prices data.

from the fourth column in **Table 6**, which favors the QL method and gives smaller standardized residuals, better than AQL method.

5. Conclusions

In this chapter, two alternative approaches, QL and AQL, have been developed to estimate the parameters in ARCH and GARCH models. Parameter estimation for ARCH and GARCH models, which include nonlinear and non-Gaussian models is given. The estimations of unknown parameters are considered without any distribution assumptions concerning the processes involved, and the estimation is based

on different scenarios in which the conditional covariance of the error's terms are assumed to be known or unknown. Simulation studies and empirical analysis show that our proposed estimation methods work reasonably quite well for parameter estimation of ARCH and GARCH models. It will provide a robust tool for obtaining an optimal point estimate of parameters in heteroscedastic models like ARCH and GARCH models.

This chapter focuses on models in univariate, while it is desirable to consider multivariate extensions of the proposed models.

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References

- [1] Engle RF. Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*. 1982;**50**:987-1008
- [2] Engle RF. GARCH 101: The use of ARCH/GARCH models in applied econometrics. *The Journal of Economic Perspectives*. 2001;**15**:157-168
- [3] Bollerslev T, Chou RY, Kroner KF. ARCH modeling in finance: A selective review of the theory and empirical evidence. *Journal of Econometrics*. 1992; **52**:5-59
- [4] Bera A, Higgins M. ARCH models: Properties, estimation and testing. *Journal of Economic Surveys*. 1993;**7**: 305-366
- [5] Bollerslev T, Engle RF, Nelson DB, Models ARCH. In: Engle RF, McFadden D, editors. *Handbook of Econometrics*. Vol. 4. Amsterdam: North-Holland; 1994. pp. 2959-3038
- [6] Diebold F, Lopez J. Modeling volatility dynamics. In: Hoover K, editor. *Macroeconometrics: Developments, Tensions and Prospects*. Boston: Kluwer Academic Press; 1995. pp. 427-472
- [7] Pagan A. The econometrics of financial markets. *Journal of Empirical Finance*. 1996;**3**:15-102
- [8] Palm F. GARCH models of volatility. In: Rao CR, Maddala GS, editors. *Handbook of Statistics*. Vol. 14. Amsterdam: North-Holland; 1996. pp. 209-240
- [9] Shephard N. Statistical aspects of ARCH and stochastic volatility models. In: Cox DR, Hinkley DV, Barndorff-Nielsen OE, editors. *Time Series Models in Econometrics, Finance and Other Fields*. London: Chapman Hall; 1996. pp. 1-67
- [10] Andersen T, Bollerslev T. ARCH and GARCH models. In: Kotz S, Read CB, Banks DL, editors. *Encyclopedia of Statistical Sciences*. Vol. 2. New York: John Wiley and Sons; 1998
- [11] Engle R, Patton A. What good is a volatility model? *Quantitative Finance*. 2001;**1**:237-245
- [12] Degiannakis S, Xekalaki E. Autoregressive conditional heteroscedasticity (arch) models: A review. *Quality Technology and Quantitative Management*. 2004;**1**: 271-324
- [13] Diebold F. The Nobel memorial prize for Robert F. Engle. *The Scandinavian Journal of Economics*. 2004;**106**:165-185
- [14] Andersen T, Diebold F. Volatility and correlation forecasting. In: Granger CWJ, Elliott G, Timmermann A, editors. *Handbook of Economic Forecasting*. Amsterdam: North-Holland; 2006. pp. 777-878
- [15] Engle RF, Gonzalez-Rivera G. Semiparametric ARCH models. *Journal of Business & Economic Statistics*. 1991; **9**(4):345-359
- [16] Li DX, Turtle HJ. Semiparametric ARCH models: An estimating function approach. *Journal of Business & Economic Statistics*. 2000;**18**(2):174-186
- [17] Linton O, Mammen E. Estimating semiparametric ARCH models by kernel smoothing methods. *Econometrica*. 2005;**73**(3):771-836
- [18] Linton OB. Semiparametric and nonparametric ARCH modeling. In: *Handbook of Financial Time Series*. Berlin Heidelberg: Springer; 2009. pp. 157-167
- [19] Su L, Ullah A, Mishra S. Nonparametric and semiparametric

- volatility models: Specification, estimation, and testing. In: Handbook of Volatility Models and Their Applications. Hoboken, New Jersey, USA: John Wiley Sons, Inc.; 2012. pp. 269-291
- [20] Alexander C. Market Models: A Guide to Financial Data Analysis. Chichester, UK: John Wiley and Sons, Ltd.; 2001
- [21] Alzghool R. Estimation for state space models: Quasi-likelihood and asymptotic quasi-likelihood approaches [Ph.D. thesis]. Australia: School of Mathematics and Applied Statistics, University of Wollongong; 2008
- [22] Enders W. Applied Econometric Time Series. Hoboken, NJ: John Wiley and Sons, Inc.; 2004
- [23] Taylor S. Asset Price Dynamics and Prediction. Princeton, NJ: Princeton University Press; 2004
- [24] Bollerslev T. Generalized autoregressive conditional heteroscedasticity. Journal of Econometrics. 1986;**31**(3):307-327
- [25] Hedy CC. Quasi-Likelihood and Its Application: A General Approach to Optimal Parameter Estimation. New York: Springer; 1997
- [26] Lin Y-X. A new kind of asymptotic quasi-score estimating function. Scandinavian Journal of Statistics. 2000; **27**:97-109
- [27] Hardle W. Applied Nonparametric Regression. Cambridge: Cambridge University Press; 1991
- [28] Alzghool R, Lin Y-X. Asymptotic Quasi-Likelihood Based on Kernel Smoothing for Nonlinear and Non-Gaussian State-Space Models. Lecture Notes in Engineering and Computer Science. London, UK: ICCSDE; 2007. pp. 926-932
- [29] Alzghool R, Lin Y-X. Parameters estimation for SSMs: QL and AQL approaches. IAENG International Journal of Applied Mathematics. 2008; **38**:34-43
- [30] Alzghool R, Lin Y-X, Chen SX. Asymptotic quasi-likelihood based on kernel smoothing for multivariate heteroskedastic models with correlation. American Journal of Mathematical and Management Sciences. 2010;**30**(1&2): 147-177
- [31] Alzghool R. Estimation for stochastic volatility model: Quasi-likelihood and asymptotic quasi-likelihood approaches. Journal of King Saud University-Science. 2017;**29**: 114-118
- [32] Alzghool R. Parameters estimation for GARCH (p,q) model: QL and AQL approaches. Electronic Journal of Applied Statistical Analysis (EJASA). 2017;**10**(1):180-193
- [33] Alzghool R, Al-Zubi LM. Semi-parametric estimation for ARCH models. Alexandria Engineering Journal. 2018;**57**:367-373
- [34] Alzghool R, Lin Y-X. Initial values in estimation procedures for state space models (SSMs). In: Proceedings of World Congress on Engineering, WCE 2011; London, UK: Newswood Limited; 2011
- [35] Zivot E, Wang J. Modeling Financial Time Series with S-PLUS. New York: Springer; 2006